An Approach to Linear and Nonlinear Heat-Transfer Problem Using a Lagrangian

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A new variational principle for the generalized theory of heat conduction with finite wave speed is considered. The corresponding transition to the classical case, when the speed of thermal disturbance is infinite, is performed by setting the relaxation time equal to zero. The concept of generalized coordinate (penetration depth) and the method of partial integration are basic tools for obtaining the approximative solutions. By means of the variational technique developed here a nonlinear boundary value problem can be reduced to an ordinary differential equation whose solution is often capable of being expressed in analytic closed form. Several examples are solved in details. The results are found to be in good agreement with those obtained by other methods.

Nomenclature

a	= thermal diffusivity
c	= heat capacity
erfc x	$= 1 - \left[\frac{2}{(\pi)^{1/2}}\right]\int_0^x e^{-\lambda^2}\,d\lambda$
I	= action integral
k	= thermal conductivity
L	= Lagrangian
N_{Pe}	= Peclet number
t	= time
T	= temperature
T^*	= image temperature
x_i; x, y, z	= rectangular coordinates
dv	= elementary volume
η	= thermal potential
θ	= dimensionless time
ν	= heat-transfer coefficient between the rod and its surroundings
τ	= relaxation time
∇^2	$= \Sigma_i \partial^2/\partial x_i^2$

1. Introduction

THE use of variational and Lagrangian methods in irreversible thermodynamics and heat-transfer problems have gained interest and importance in recent years.

Two of the principal reasons variational theory has been useful in heat flow problem are a) it enables one to overcome the difficulties of nonlinear problems in this area and b) it is often extremely effective in obtaining approximate solutions.

Although very successful and useful, the variational method in heat flow analysis differs from the usual variational principles of ordinary mechanics. In order to underline this fact, we will briefly review three pertinent variational principles.

In the Glansdorff-Prigogine-Schechter principle of local potential[1–3] there are two kinds of physical variables (temperature T, velocity v, pressure p etc.) the so-called thermodynamic variables which change during the process of variation on an integral and the variables of the same type (T_0, v_0, p_0, etc.) which are held fixed. This dual personality of variables must be maintained until the process of variation is complete. After that putting $T_0 = T$, $v_0 = v$, $p_0 = p$ etc., one obtains the correct differential equations of the process in consideration.

To be more explicit let us consider the Lagrangian

$$L = T(\partial T_0/\partial t) + (a/2)(\partial T/\partial x)^2, \quad a = \text{const} \tag{1}$$

Applying the Euler-Lagrange equation

$$\partial L/\partial T - (\partial/\partial x)\partial L/\partial(\partial T/\partial x) - (\partial/\partial t)\partial L/\partial(\partial T/\partial t) = 0$$

and considering the term $\partial T_0/\partial t$ to be constant one gets

$$\partial T_0/\partial t = a(\partial^2 T/\partial x^2)$$

This equation together with the subsidiary condition $T_0 = T$ is the correct differential equation of transient heat conduction in the absence of heat sources. In the sense of the variational principles of classical mechanics, this separation of variables is not quite correct. But the method cited has great engineering significance in many practical applications.

The difference between mechanical variational principles and variational principles in heat-transfer analysis becomes more evident in the case of the hydrodynamical Hamiltonian principle stated by Bateman.[4] This principle was used and elaborated on by Morse and Feshbach[5] and Slattery.[6]

Let us consider the Lagrangian

$$L = (\tfrac{1}{2})[T^*(\partial T/\partial t) - T(\partial T^*/\partial t)] + a(\partial T/\partial x)(\partial T^*/\partial x) \tag{2}$$
$$a = \text{const}$$

where T is the real temperature and T^* is the "image temperature" or associated function without any physical interpretation.

Applying the Lagrangian equations:

$$\partial L/\partial T - (\partial/\partial t)\partial L/\partial(\partial T/\partial t) - (\partial/\partial x)\partial L/\partial(\partial T/\partial x) = 0$$
$$\partial L/\partial T^* - (\partial/\partial t)\partial L/\partial(\partial T^*/\partial t) - (\partial/\partial x)\partial L/\partial(\partial T^*/\partial x) = 0$$

we obtain two differential equations of the form

$$\partial T/\partial t = a(\partial^2 T/\partial x^2) \tag{3}$$

$$\partial T^*/\partial t = -a(\partial^2 T^*/\partial x^2) \tag{4}$$

Differential Eq. (3) is correct equation of physical process but differential Eq. (4) obviously does not have any physical significance. Consequently, there are no guides in picking boundary and initial conditions for the auxiliary variable T^*.

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Another variational approach to heat-transfer analysis is Biot's variational principle.[7-10] Using the concepts of thermal potential, dissipation function, and generalized thermal forces, and with the help of generalized coordinates a very powerful technique for solving nonlinear boundary value problems was established. The characteristics of Biot's analysis is that the Lagrangian does not exist, i.e., the principle does not have the Hamiltonian structure as in the two previously mentioned principles.

In the present paper, we shall consider a new Lagrangian for a generalized theory of heat conduction with finite wave speed. The corresponding transition to the classical case when the speed of thermal disturbance is infinite, is performed by setting the relaxation time equal to zero. Using the standard variational technique in the form of partial integration, it is possible to obtain the solutions of some nonlinear engineering problems without linearizing the equations. The notion of penetration depth introduced by Biot is employed throughout.

2. Formulation of the Theory

The governing equation for intensive nonstationary temperature field in linear theory of heat flow is[11-13]

$$c\tau(\partial^2 T/\partial t^2) + c(\partial T/\partial t) = k\nabla^2 T \tag{5}$$

where T is the temperature, τ relaxation time, c heat capacity, k thermal conductivity.

Equation (5) may be derived from a variational principle

$$\delta\int\int L[t,T,(\partial T/\partial t),(\partial T/\partial x_i)]dvdt$$

where the Lagrangian is

$$L = [(c\tau/2)(\partial T/\partial t)^2 - (k/2)\ \Sigma_i\ (\partial T/\partial x_i)^2]e^{t/\tau} \tag{6}†$$

The Euler-Lagrange equation

$$\partial L/\partial T - \frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial T/\partial t)} - \Sigma_i \frac{\partial}{\partial x_i}\frac{\partial L}{\partial(\partial T/\partial x_i)} = 0$$

gives Eq. (5).

In this article we will focus our attention on the classical case, i.e., when the relaxation time τ in Eq. (5) tends to zero. In order to get correct results using the Lagrangian Eq. (6) we must obey the simple rule that after completing the variational process (and dividing with $e^{t/\tau}$) we will apply the subsidiary condition $\tau \rightarrow 0$. This process is formally very close to the Glansdorff-Prigogine principle and it has obvious physical meaning.

In the case of temperature, dependent thermal properties (nonlinear case), Likov and Mikhailov (Ref. 14, p. 485), have shown that the general equation of nonlinear heat and mass flow may be written in the form

$$F(\eta)(\partial\eta/\partial t) = \nabla^2\eta \tag{7}$$

where η is a potential depending on the temperature.

Let us consider the Lagrangian

$$L = [(\tau/2)F(\eta)(\partial\eta/\partial t)^2 - (\tfrac{1}{2})\Sigma_i(\partial\eta/\partial x_i)^2]e^{t/\tau} \tag{8}$$

If we apply the Euler-Lagrange equation, after dividing by $e^{t/\tau}$ and requiring that $\tau \rightarrow 0$ we will get Eq. (7). Hence, a general variational approach to linear and nonlinear heat flow analysis is established.

Since the variational principle Eqs. (6) and (8) does not introduce any boundary conditions, a method for including natural and flux boundary conditions is necessary. The method to be employed here is identical to the method of Biot[7] and Lardner.[8] In many physically important situations, it is possible to select the trial solution in a form so that the corresponding boundary conditions are identically satisfied and explicit use of the boundary conditions is unnecessary.

† For the corresponding Lagrangian in classical mechanics and vibration theory see Ref. 15.

3. Applications to Some One-Dimensional Problems

The one-dimensional problem, because of its relative simplicity, has had a broad treatment in the literature. The boundary and initial conditions considered are the same as in well-known works by Schechter[3] and Biot,[7] whose results are used for comparison with present results.

Temperature Distribution in a Finite Insulated Rod with Constant Parameters

Let us consider a thermally insulated rod of $2l$ length. The ends of the rod are maintained at a constant temperature $T = 0$. Assume that the initial temperature distribution is given by

$$T_0[1 - (x/l)^2]$$

where T_0 is a constant. The problem is to find the temperature as a function of position and time, which is a solution of the boundary value problem,

$$\partial T/\partial t = a(\partial^2 T/\partial x^2)$$

Subject to the initial and boundary conditions

$$T = T_0\ [1-(x/e)^2] \text{ at } t = 0 \text{ for } -l \leq x \leq l$$

$$T = 0 \text{ at } x = \pm l \quad \text{for } t \geqslant 0$$

where a is the thermal diffusivity.

Let us consider the action integral

$$I = \int_{t_0}^{t_1}\int_{-l}^{l}\left[(\tau/2)\left(\frac{\partial T}{\partial t}\right)^2 - \left(\frac{a}{2}\right)\left(\frac{\partial T}{\partial x}\right)^2\right]e^{t/\tau}\,dxdt \tag{9}$$

The time interval $t_0 \rightarrow t_1$ is chosen arbitrarily.

In order to obtain an approximate solution, we will apply the method of partial integration. Assume that the temperature distribution is of the form

$$T = T_0[1 - (x/l)^2]f(t);\ f(0) = 1 \tag{10}$$

where $f(t)$ is an unknown function of the time only. Substituting Eq. (10) into Eq. (9) and integrating with respect to x we have

$$I = \int_{t_0}^{t_1}\left[\left(\frac{\tau}{2}\right)(\tfrac{16}{15})l\dot{f}^2 - (\tfrac{4}{3})\left(\frac{a}{l}\right)f^2\right]e^{t/\tau}\,dt \equiv \int_{t_0}^{t_1} L(f,\dot{f},t)dt \tag{11}$$

where a constant multiplicative factor is omitted. In order to assure that the action integral in Eq. (11) has a stationary value the Euler-Lagrange equation

$$(d/dt)(\partial L/\partial\dot{f}) - \partial L/\partial f = 0$$

must be satisfied.

Hence we have

$$(\tfrac{16}{15})\tau l\ddot{f} + (\tfrac{16}{15})l\dot{f} + (\tfrac{8}{3})(a/l)f = 0$$

Putting $\tau \rightarrow 0$ we have the ordinary differential equation

$$(\tfrac{16}{15})l\dot{f} + (\tfrac{8}{3})(a/l)f = 0$$

Integrating, since $f(0) = 1$,

$$f = e^{-(5/2)(a/l^2)t}$$

Hence,

$$T = T_0[1 - (x/l)^2]e^{(5/2)(a/l^2)t}$$

The same result was obtained and compared with the exact solution in Ref. 3, using the Glansdorff-Prigogine variational method.

Noninsulated Rod—Exponential Profile

The heat conduction equation for a noninsulated rod of constant cross section is of the form

$$\partial T/\partial t = a(\partial^2 T/\partial x^2) - \nu T$$

where ν is related to the heat-transfer coefficient between the rod and its surroundings.

Let us suppose that one face of the rod at $x = 0$ is heated suddenly to the temperature T_0 at $t = 0$. We state the temperature T_0 as a reference temperature in the problem and introduce the generalized coordinate $q(t)$ in the form used by Goodman:

$$T = T_0 e^{-x/q(t)} \tag{12}$$

Let us consider the action integral of the form

$$I = \int_{t_0}^{t_1}\int_0^{\infty}\left[\left(\frac{\tau}{2}\right)\left(\frac{\partial T}{\partial t}\right)^2 - \left(\frac{\nu}{2}\right)T^2 - \left(\frac{a}{2}\right)\left(\frac{\partial T}{\partial x}\right)^2\right] \times e^{t/\tau}\,dxdt \tag{13}$$

Substituting Eq. (12) into Eq. (13) and integrating with respect to x one has

$$I = \int_{t_0}^{t_1}\left(\frac{\tau}{8}\right)\left(\frac{\dot{q}^2}{q}\right) - \left(\frac{\nu q}{4}\right) - \frac{a}{4q}\Big]e^{t/\tau}\,dt \equiv \int_{t_0}^{t_1} L(q,\dot{q},t)dt$$

where a constant multiplicative factor is omitted. Applying the Euler-Lagrange equation we have the differential equation

$$(\tau/4)(d/dt)(\dot{q}/q) + (\tau/8)(\dot{q}^2/q^2) + (\tfrac{1}{4})(\dot{q}/q) + \nu/4 - a/4q^2 = 0$$

For $\tau \to 0$ we have

$$\dot{q} + \nu q - a/q = 0$$

the solution of which along with the initial condition $q(0) = 0$ is

$$q = (a/\nu)^{1/2}(1 - e^{-2\nu t})^{1/2}$$

The temperature gradient at $x = 0$ is

$$\partial T/\partial x = -T_0(\nu/a)^{1/2}/(1 - e^{-2\nu t})^{1/2}$$

The same result was obtained by Goodman in Ref. 16 (pp. 66–67) through the help of integral method. For comparison of this result with the exact solution see the same reference.

Heating of a Slab with Temperature Dependent Parameters

As an example, we study the heating of a homogeneous slab whose capacity c is a function of the temperature, and the conductivity k is constant.

Let us assume that the capacity is of the form

$$c = c_0[1 + (T/T_0)] \tag{14}$$

Again using an approximation similar to Biot's let us assume

$$T = T_0(1 - x/f)^2 \tag{15}$$

where $f(t)$ is depth of penetration of the temperature rise.

Consider the action integral

$$I = \int_{t_0}^{t_1}\int_0^{f}\left[\left(\frac{\tau}{2}\right)c(T)\left(\frac{\partial T}{\partial t}\right)^2 - \left(\frac{k}{2}\right)\left(\frac{\partial T}{\partial x}\right)^2\right]e^{t/\tau}\,dxdt \tag{16}$$

Substituting Eqs. (14) and (15) into Eq. (16) and integrating with respect to x, this expression reduces to

$$I = \int_{t_0}^{t_1}\left[(\tfrac{3}{70})\,\tau c_0\left(\frac{\dot{f}^2}{f}\right) - \frac{k}{3f}\right)\Big]e^{t/\tau}\,dt$$

where a constant multiplicative factor is omitted. The Euler-Lagrange equation yields

$$(\tfrac{3}{70})\tau c_0(\dot{f}/f)^2 + (\tfrac{3}{35})\tau c_0(d/dt)(\dot{f}/f) + (\tfrac{3}{35})c_0(\dot{f}/f) - k/3f^2 = 0 \tag{17}$$

Putting $\tau \to 0$ and solving the remaining equation of the first order, with initial condition $f(0) = 0$, we find

$$f = 2.80(kt/c_0)^{1/2} \tag{18}$$

Biot has found for the same problem

$$f = 2.97(kt/c_0)^{1/2}$$

A Problem

As our last example, we shall study the rate of heat or mass transfer to a fluid in ideal stagnation flow toward a flat interface. According to Ref. 3, the differential equation of this process is of the form

$$\partial T/\partial\theta - 2N_{Pe}y(\partial T/\partial y) = \partial^2 T/\partial y^2 \tag{19}$$

Where N_{Pe} is the Peclet number (a constant), T is the temperature or concentration, θ is the dimensionless time and y is the dimensionless coordinate normal to the interface.

The initial and boundary conditions are

$$T = 0 \text{ at } \theta = 0 \tag{20}$$

$$T = 1 \text{ at } y = 0 \tag{21}$$

$$T \to 0 \text{ as } y \to \infty \tag{22}$$

Let us consider the action integral of the form

$$I = \int_{t_0}^{t_1}\int_0^{\infty}\left[\left(\frac{\tau}{2}\right)\left(\frac{\partial T}{\partial\theta}\right)^2 - (\tfrac{1}{2})\left(\frac{\partial T}{\partial y}\right)^2\right]e^{N_{Pe}y^2}e^{\theta/\tau}dtd\theta \tag{23}$$

It is easy to verify that Euler-Lagrange equation leads to Eq. (19) with the subsidiary condition $\tau \to 0$.

Assuming a solution of the form

$$T = \mathrm{erfc}[f(\theta)y] \tag{24}$$

the boundary conditions in Eq. (21) and (22) are satisfied automatically and the initial condition in Eq. (20) is satisfied if

$$f(\theta) \to \infty \quad \text{as } \theta \to 0 \tag{25}$$

From Eq. (24) one has

$$\partial T/\partial\theta = -[2/(\pi)^{1/2}]\dot{f}_y e^{-f^2y^2};\quad \dot{f} \equiv df/d\theta$$

$$\partial T/\partial y = -[2/(\pi)^{1/2}]fe^{-f^2y^2}$$

Putting these relations into Eq. (23)

$$I = \int_{t_0}^{t_1}\left\{\left[\left(\frac{\tau}{2}\right)\frac{\dot{f}^2}{(f^2 - N_{Pe}/2)^{3/2}}\int_0^{\infty}\lambda^2 e^{-2\lambda^2}d\lambda\right] - \left[\frac{f^2}{(f^2 - N_{Pe}/2)^{1/2}}\int_0^{\infty}e^{-2\lambda^2}d\lambda\right]\right\}e^{\theta/\tau}d\theta$$

where a constant multiplicative factor is omitted and

$$\lambda^2 = (f^2 - N_{Pe}/2)y^2$$

Using the well-known relations

$$\int_0^\infty \lambda^2 e^{-2\lambda^2}\, d\lambda = \frac{(\pi)^{1/2}}{16}$$

$$\int_0^\infty e^{-2\lambda^2}\, d\lambda = \frac{(\pi)^{1/2}}{4}$$

we have

$$I = \int_{t_0}^{t_1} \left[\left(\frac{\tau}{2}\right)\left(\frac{1}{4}\right) \frac{\dot{f}^2}{(f^2 - N_{Pe}/2)^{3/2}} - \left(\frac{1}{2}\right) \frac{f^2}{(f^2 - N_{Pe}/2)^{1/2}} \right] e^{\theta/\tau}\, d\theta$$

where the constant multiplicative factor $(\pi)^{1/2}/4$ is omitted.

Hence, Euler-Lagrange equation yields

$$\left(\frac{3}{8}\right)\frac{f\dot{f}^2}{(f^2 - N_{Pe}/2)^{5/2}} + \left(\frac{\tau}{4}\right)\left(\frac{d}{d\theta}\right)\left[\frac{\dot{f}}{(f^2 - N_{Pe}/2)^{3/2}}\right] + \left(\frac{1}{4}\right)\frac{\dot{f}}{(f^2 - N_{Pe}/2)^{3/2}} + \frac{f^3 - fN_{Pe}}{2(f^2 - N_{Pe}/2)^{3/2}} = 0$$

For $\tau \to 0$ we have the ordinary differential equation:

$$\dot{f} - 2N_{Pe}f + 2f^3 = 0$$

Solving for f we find

$$f = [(1/N_{Pe})(1 - e^{-4N_{Pe}\theta})]^{-1/2}$$

and the initial condition in Eq. (25) is satisfied. Substituting this into the trial function in Eq. (24) we finally have

$$T = \operatorname{erfc}[(1/N_{Pe})(1 - e^{-4N_{Pe}\theta})]^{-1/2}$$

This exact solution has been found by Chan; the same solution was obtained and discussed by the help of Glansdorff-Prigogine principle in (Ref. 3, pp. 262–267). The exactness of the solution obtained is merely the result of a very fortunate choice of expression in Eq. (24).

4. Discussion

It has been shown in Ref. 3 (p. 255) that a variational principle for the classical heat conduction problem (infinite speed of thermal disturbance) does not exist. Using the generalized heat conduction theory with finite wave speed, it is possible to obtain a new variational technique obtain an approximate solution for the classical problem. Agreement of the present method with the known solutions of various problems is quite satisfactory.

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